This is a brief description of sage code relating to types and the Kac-Stanley formula.

The file `http://www.imsc.res.in/~amri/types3.sage` is the code on types. Recall that the similarity classes of $n \times n$ matrices with entries in $\mathbf{F}_q$ are parametrized by functions

$$c : \operatorname{Irr} \mathbf{F}_q[t] \to \Lambda \tag{1}$$

(Λ denotes the set of all partitions) such that

$$\sum_{f \in \operatorname{Irr} F_q[t]} |c(f)| \deg f = n.$$

The latter condition implies that $c(f) = \emptyset$ ($\emptyset$ is the empty partition, a partition of 0). Similarity classes corresponding to c and c' are said to be of the same type if there exists a degree-preserving bijection $\phi : \operatorname{Irr} \mathbf{F}_q[t] \to \operatorname{Irr} \mathbf{F}_q[t]$ such that $c' = c \circ \phi$. The nice thing about types is that:

(1) The number of types is independent of q.
(2) One can easily calculate the number of classes of each type.
(3) All classes of the same type have the same centralizer in $GL_n(\mathbf{F}_q)$ and hence also the same cardinality, which can be computed from the type.

In order to clarify what is meant by *computed from the type*, it is necessary to explain how a type is represented in a computer program. Given a function c as in (1), for each $f \in \operatorname{Irr} \mathbf{F}_q[t]$, write down the pair $(c(f), \deg f)$. To a type, we can associate a multiset

$$(\lambda^{(1)}, d_1)^{a_1}, (\lambda^{(2)}, d_2)^{a_2}, \dots) \tag{2}$$

where a_i is the number of irreducible polynomials $f \in \operatorname{Irr} \mathbf{F}_q[t]$ of degree d_i for which $c(f) = \lambda^{(i)}$. Types of $n \times n$ matrices are in bijective correspodence with multisets where

$$\sum_i |\lambda^{(i)}| d_i a_i = n.$$

This is the data that is used to construct the class `matrix_type`.

An application is the calculation of the number of simultaneous similarity classes of k-tuples of $n \times n$ matrices: consider the set $M_n(\mathbf{F}_q)^k$ on which $GL_n(\mathbf{F}_q)$ acts by $g \cdot (A_1, \dots, A_k) = (gA_1a^{-1}, \dots, gA_kg^{-1})$. Burnside's lemma allows us to calculate the number of classes:

$$GL_n(\mathbf{F}_q) \backslash M_n(\mathbf{F}_q)^k = \frac{1}{|GL_n(\mathbf{F}_q)|} \sum_{g \in G} |Z_{GL_n(\mathbf{F}_q)}(g)|.$$

This is implemented using types as follows:

```
sage: load ``http://www.imsc.res.in/~amri/types3.sage''
sage: def A(n,k):
...       return sum([tau.matrices(invertible = True)*q^(tau.z()*k)
for tau in matrix_types(n)])/GL_field(n)
sage: A(2,2)
q^5 + q^4 + q^3
```

The function `A(n,k)` defined here returns the number of similarity classes of k-tuples of $n\times n$ matrices. It is easy to read off from a matrix type whether or not it is regular, semisimple, etc., so this code can be used to enumerate the number of matrices in $M_n(\mathbf{F}_q)$ or $GL_n(\mathbf{F}_q)$ which are regular, semisimple, etc. For example

```
sage: def ss(n, invertible = False):
...       return sum([tau.matrices(invertible = invertible) for tau
in filter(lambda tau: tau.is_semisimple(), matrix_types(n))])
sage: ss(2)
q^4 - q^3 + q
sage: ss(2, invertible = True)
q^4 - 2*q^3 + 2*q - 1
```

In `http://www.imsc.res.in/~amri/multitypes2.sage` , the idea behind types is extended to quivers. Recall that for a quiver Q with vertex set $\{1,\dots,n\}$ and edge set E, a representation is a collection of vector spaces $V_1,\dots,V_n$, and a collection $T_e : V_{s(e)} \to V_{t(e)}$ of linear maps. Here $s(e)$ and $t(e)$ denote the source and target of e. The dimension vector of the representation is the vector $(\dim(V_1),\dots,\dim(V_n))$. Isomorphism classes of representations with dimenstion vector $v = (v_1,\dots,v_n)$ correspond to orbits for the action of $\prod_{i=1}^n GL_{v_i}(\mathbf{F}_q)$ on $\prod_e M_{v_{t(e)}\times v_{s(e)}}(\mathbf{F}_q)$. The Kac-Stanley formula [2] adapts the theory of types to count the number of such orbits. The only difference between a type and a multitype is that whereas a type is a multiset of pairs (λ, d) where λ is partition, a multitype is a multiset of pairs (λ, d) where λ is a vector partition (an unordered tuple of vectors with non-negative integer entries).

By knowing the total number of isomorphism classes of representations of a quiver for various dimension vectors, it is possible to compute the number of isomorphism classes of indecomposable or absolutely indecomposable representations. The integrality of the coefficients of the latter (which is called the Kac polynomial) was a recent breakthrough Hausel, Lettelier and Rodriguez-Villegas [1]. **Cycles and loops are allowed for these algorithms**.

```
sage: load "http://www.imsc.res.in/~amri/multitypes2.sage"
sage: Q = graphs.CompleteGraph(3)
sage: representations(Q, [1, 1, 1])
q + 6
sage: representations(Q, [1, 1, 1], indecomposable = True)
q + 2
sage: KacPolynomial(Q, [1, 1, 1])
q + 2
```

References

[1] T. Hausel, E. Letellier, and F. Rodriguez-Villegas. Positivity of Kac polynomials and DT-invariants for quivers. *Annals of Mathematics*, 177:1147–1186, 2013.

[2] V. G. Kac. Root systems, representations of quivers and invariant theory. In *Invariant theory*, pages 74–108. Springer, 1983.